NORMAL FORMS FOR ONE- AND TWO-SIDED SURFACES.

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1. In view of the difficulties that are still to be overcome before normal forms can be obtained for manifolds of more than two dimensions, a simplified treatment of the analogous problem for closed surfaces may be worth mentioning. We shall prove that all one-sided surfaces of a given connectivity k are topologically equivalent and can therefore be reduced to a single normal form; also, that the same is true of all closed surfaces of a given genus $p = (k - 1)/2$.* Wherever the reasoning is of an intuitional character, it may readily be put into perfectly rigorous form by means of known theorems. We shall assume that the connectivity of a surface, as defined by the generalized Euler relation

$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - k,\dagger$$

is finite and invariant, and furthermore that the sphere is a normal form to which every closed surface of connectivity 1 may be reduced.

2. Upon any closed surface S, let us describe as many non-intersecting closed paths as possible without separating the surface into distinct regions. According to the assumptions made in § 1, the number of these paths is finite. Then if we cut the surface S along the paths, there will remain a surface S' bounded by certain curves which we shall suppose to be $k - 1$ in number. The points of these boundary curves will be included in the surface S' so that there will really be two points of S' to every point of a cut in S.

It can easily be shown that the surface S' is equivalent to a sphere from which $k - 1$ simply connected regions have been removed. For if we were to match the edge of a simply connected bounded surface along each of the bounding curves of S', we should obtain a closed surface S'' consisting of S' and the adjoined surfaces. Moreover, the surface S'' would surely be equivalent to a sphere, otherwise a closed cut could be made in S'' without severing it. But if this were possible, the cut could always be made in such a way as not to pass through the adjoined surfaces,

* A proof of this theorem is sketched in the article on Analysis Situs by Dehn and Heegaard in the Encyklopädie der mathematischen wissenschaften where references to the literature will be found.

† In this equation α_0, α_1 and α_2 represent the number of vertices, edges, and faces respectively of an arbitrary decomposition of the surface into simply connected pieces.

the latter being simply connected. We should thus be able to make a closed cut in S' without severing it, contrary to hypothesis. The surface S' will for convenience be regarded as a Riemann plane from which the interiors of $k - 1$ circles have been removed.

3. Now two classes of closed cuts may be distinguished on a surface. If we start at a point P on one edge of the cut and describe a path which follows along the cut without crossing it, we shall return, after having made a complete circuit, either to the point P or to the point P' homologous to P but on the other edge of the cut. When the first of the two possibilities is realized, the cut is said to be two-sided; in the contrary case, it is said to be one-sided. A two-sided cut gives rise to two boundary curves which are in point-for-point continuous correspondence with one another; a one-sided cut gives rise to a single curve only, consisting of two segments ABC and CDA which are in point-for-point continuous correspondence with one another.

4. Returning to the representation S' of S in the Riemann plane, we shall distinguish three types of bounding circles.

I. Circles arising from one-sided cuts, opposite points of which (i. e. points at the extremities of the same diameter) correspond to the same point of S.

II. Pairs of circles arising from two-sided cuts and such that when a point P be made to describe one of the cuts, the two images of P in S' always describe their respective circles in the same sense.

III. Pairs of circles arising from two-sided cuts and such that when a point P be made to describe one of the cuts, the two images of P in S' describe their respective circles in opposite senses.

5. When and only when the circles of the surface S' are all of type III, the surface S is two-sided. In this case, S' can readily be reduced to any one of the well-known normal forms. Thus, Klein's normal form, the sphere with p handles, is obtained by deforming the surface S' in such a way that each circle becomes the extremity of a tubular neck protruding from the body of the surface. By joining corresponding circles to one another, each pair of tubular necks will give rise to a handle.

Another convenient normal form for a two-sided surface other than a sphere is a plane bounded by p rectangles, where opposite points of a rectangle correspond to the same point of S. This form can be obtained at once if we cut the surface S' along a set of non-intersecting curves, one for each pair of circles, and such that each curve joins a point of one circle to the point on the other circle of the pair which represents the same point of S. For every curve of the new boundary can then be deformed into a rectangle, two of whose sides will correspond to a pair of circles, while the remaining two will correspond to the edges of a cut.

Finally, if we deform all of the rectangles so as to make a corner of each pass through the same point P, we shall have another well-known form, a simply connected surface bounded by a chain of arcs

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}, \quad \cdots, \quad a_pb_pa_p^{-1}b_p^{-1}.*$$

which may be deformed into a polygon.

6. The model S' of a one-sided surface S contains at least one circle of Type I, or else a pair of circles of Type II. Let us first observe that

A pair of circles of Type II can be replaced by two circles of Type I.

Join the points P and Q of the first circle to the two corresponding points P' and Q' respectively of the second by means of two simple non-intersecting curves p and q respectively. Cut the surface S' along the lines p and q, thereby severing from S' a piece E bounded by one edge of the cut p, an arc r of the first circle, one edge of the cut q, and an arc s of the second circle (Fig. 1).

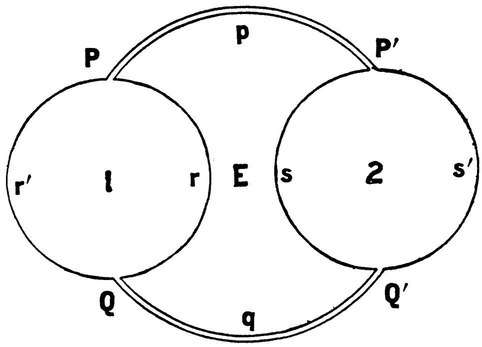

Fig. 1.

Now the arc r of the first circle measured from P to Q corresponds to the same points of the surface S as the arc s' complementary to s on the second circle measured from P' to Q'. Similarly, the arc s measured from P' to Q' corresponds to the same set of points as the arc r' complementary to r measured from P to Q. We may therefore re-affix the element E to the surface by matching s against r' and r against s'. The openings which corresponded to the circles are now closed up and there remain two other openings, one bounded by both edges of the cut q, the other by both edges of the cut p. The boundary of each of these openings is equivalent to a circle of Type I.

Furthermore,

* The symbols a_i and a_i^{-1} denote the same arc, but in one case, the arc is described in one sense, in the other case, in the other.

A pair of circles of Type III may be replaced by a pair of Type II, provided there exists at least one circle of Type I.

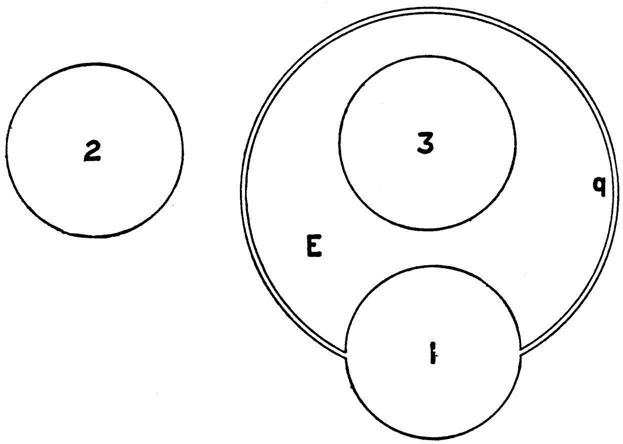

Fig. 2.

For let 1 be a circle of Type I, and 2 and 3, a pair of circles of Type III. If a cut q be made joining two opposite points of 1, a piece E of the surface will be severed from the rest. The surface E may be rejoined by matching the arc of the circle 1 belonging to E with the complementary arc, since the two arcs correspond to the same points of S. The circle 1 is thereby replaced by a curve composed of the two edges of q and equivalent to a circle of Type I. But the surface E must have been replaced with its lower and upper faces interchanged, since opposite points of the circle 1 corresponded to one another. Therefore, the sense of any closed curve traced upon E must have been reversed by the transformation. If then the cut q be chosen so that the surface E contains the circle 3 but not the circle 2, the pair will be transformed from one of Type III to one of Type II.

By means of the above two theorems, the diagram S' of a one-sided surface can always be reduced to a plane surface bounded by k–1 circles of Type I. Another normal form can be obtained by deforming the k–1 circles into curves passing through a common point P. Remembering that each circle may be subdivided into two arcs which correspond to the same arc of the surface S, we shall then have the one-sided surface S represented by a simply connected surface bounded by a chain of arcs

$$a_1 a_1 a_2 a_2 \cdots a_{k-1} a_{k-1}$$

which may be deformed into a polygon.

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